SPECTRAL GEOMETRY · NUMERICAL STUDY · JANOS GABOR MELEGH

INTERIM PUBLICATION · ONGOING RESEARCH

Spectral Green-Function Analysis: Two-Regime Structure, Finite-Size Scaling, and Inner Core Behavior

Numerical evidence for two-sector structure in discrete lattice Green functions across dimensions d = 2, 3, and 4. The study covers 462 total runs and provides a complete characterization of both the inner physical regime and the outer finite-size boundary tail.

Dimensions: d = 2, 3, 4 **Total runs:** 168 (main) + 294 (scaling test) **Date:** March 2026

Status: Interim — not a final closed-form claim

ABSTRACT

We report interim numerical results for the spectral Green function of a regularized Poisson kernel on discrete lattices in dimensions d = 2, 3, and 4. A single global power law is insufficient to describe the field across its full radial range in every explored parameter combination (168/168 runs). Instead, the data consistently support a two-regime decomposition: an inner sector whose exponent approaches the classical expectation d − 2, and an outer sector whose position scales linearly with system size, identifying it as a finite-size boundary tail rather than an independent physical regime. In d = 4, the inner exponent extrapolates to $a_\infty \approx 2.05$, within 2.5% of the classical value d − 2 = 2. In d = 3, convergence toward d − 2 = 1 is visible but not yet complete. In d = 2, a log-corrected power law provides the best description, consistent with d = 2 being a

marginal dimension. These results should be regarded as current research-stage conclusions rather than final closed-form claims.

CONTENTS

1. Motivation and scope
2. Generator and computational setup
3. Two-regime structure: main scan results
4. Inner core regime and N-extrapolation
5. d = 2: log-corrected power law
6. Breakpoint scaling: finite-size boundary tail
7. Outer sector: a_2 stability analysis
8. Bins robustness
9. Summary and interpretation
10. Limitations and next steps

1. Motivation and scope

The central question of this study is whether the spectral Green function of a discrete lattice Poisson kernel is adequately described by a single global power law, or whether distinct scaling regimes emerge that require separate characterization. This question is motivated by discrete relational network models in which effective geometry is expected to emerge from two-point spectral structure rather than from a primitive continuum manifold.

A secondary question concerns the physical status of any apparent second regime: is it a genuine new physical scaling law, or a finite-size artifact induced by the finite computational domain? Both questions are addressed explicitly with numerical evidence below.

2. Generator and computational setup

The numerically studied object is the regularized spectral Poisson Green function defined in Fourier space:

> GENERATOR
>
> $G(k) = 1 / (|k|^2 + \varepsilon)$, zero mode removed

A delta source is placed at the center of a d-dimensional periodic lattice of linear size N. After inverse FFT, the real-space field $\phi(r)$ is radially binned using logarithmic bins. The regularization parameter ε controls the infrared behavior; results with $\varepsilon = 0.01$ are excluded from physical conclusions due to anomalous numerical behavior documented below. Three competing functional forms are then fit against the binned (r, ϕ) data and compared via AIC.

PARAMETER SPACE

PARAMETER	VALUES EXPLORED	NOTES
d	2, 3, 4	Spatial dimension
N	48, 64, 96, 128, 160	d=4 capped at N=128 by memory
ε	10^{-3}, 10^{-4}, 10^{-5}	ε=0.01 excluded from conclusions
bins (main scan)	40, 50, 60	168 total runs
bins (scaling test)	40, 60, 80, 100, 120, 160, 200	294 additional runs

MODELS

MODEL	FUNCTIONAL FORM	FREE PARAMETERS
Single power law	$\phi \sim C + A\, r^{-\alpha}$	3

MODEL	FUNCTIONAL FORM	FREE PARAMETERS
Two-sector power law	Two power laws split at best-AIC breakpoint r^*	7
Pure log (d=2)	$\phi \sim C + A\ (\log r)^{\gamma}$	3
Log-corrected power (d=2)	$\phi \sim C + A\ r^{-\alpha}\ (\log r)^{B}$	4

3. Two-regime structure: main scan results

The two-sector model outperforms the single power law in every valid run across all dimensions, system sizes, and regularization parameters explored.

D = 2 TWO-SECTOR WINS
60 / 60
100% of runs

D = 3 TWO-SECTOR WINS
60 / 60
100% of runs

D = 4 TWO-SECTOR WINS
48 / 48
100% of runs

TOTAL
168 / 168
Single law never won

SUMMARY STATISTICS ACROSS ALL RUNS

DIM.	SINGLE A	A_1 (INNER)	A_2 (OUTER)	SPLIT_R / N	AIC GAIN	D − 2
d = 2	1.261 ± 0.217	0.766 ± 0.149	5.265 ± 1.045	0.168 ± 0.028	67.7 ± 12.0	0
d = 3	1.844 ± 0.215	1.374 ± 0.150	6.395 ± 1.183	0.213 ± 0.036	79.8 ± 16.4	1
d = 4	2.663 ± 0.198	2.204 ± 0.121	6.305 ± 1.762	0.264 ± 0.052	98.6 ± 41.7	2

The single-fit exponent consistently overestimates the inner exponent because the steep outer tail pulls the global fit upward. The AIC gain of 68–99 across dimensions is robust and not attributable to bins choices (see Section 8).

4. Inner core regime and N-extrapolation

The inner exponent a_1 exhibits a mild downward drift with increasing N, reflecting the influence of the outer tail on the two-sector fit at smaller system sizes. To estimate the asymptotic value a_∞, the functional form $a_1(N) = a_\infty + c / N^\beta$ is fit for each (d, ε) combination via grid search over β.

CORE-ONLY EXPONENTS (OUTER TAIL EXCLUDED AT 10/20/30%)

DIM.	CORE A (10%)	CORE A (20%)	CORE A (30%)	D − 2
d = 2	0.946	0.815	0.721	0
d = 3	1.506	1.385	1.302	1
d = 4	2.346	2.226	2.156	2

Progressively excluding more of the outer tail brings the core exponent steadily closer to the classical target, most clearly in d = 4.

EXTRAPOLATED A_∞ FROM $A_1(N)$ FITS ($E \leq 10^{-3}$ ONLY)

D	$E = 10^{-3}$	$E = 10^{-4}$	$E = 10^{-5}$	MEAN	D − 2
d = 2	0.615	0.367	0.309	**0.430**	0
d = 3	1.278	1.071	1.031	**1.127**	1
d = 4	2.153	2.015	1.975	**2.048**	2

KEY RESULT

In d = 4, the extrapolated values form a convergent sequence (2.153 → 2.015 → 1.975) as ε decreases. The asymptotic value $a_\infty \approx 2.05$ is within 2.5% of the classical expectation d − 2 = 2. This is the strongest quantitative result of the study. In d = 3, the sequence (1.278 → 1.071 → 1.031) shows clear but incomplete convergence toward d − 2 = 1. In d = 2, a_∞ does not converge toward 0, consistent with the log-corrected nature of the d = 2 case (Section 5).

5. d = 2: log-corrected power law

The classical expectation for d = 2 is d − 2 = 0, which implies logarithmic rather than pure power-law behavior. To test this, three models were directly compared on identical ϕ(r) data for all 120 tested d = 2 cases.

MODEL	FUNCTIONAL FORM	WINS (OF 120)
Pure power law	$\phi \sim C + A\, r^{-\alpha}$	0
Pure logarithm	$\phi \sim C + A\, (\log r)^{\gamma}$	0
Log-corrected power	$\phi \sim C + A\, r^{-\alpha}\, (\log r)^{B}$	**120 / 120**

KEY RESULT

The log-corrected power law is the best-fitting model in all 120 tested d = 2 cases, across all N, ε, and bins combinations. The d = 2 field is therefore neither a clean power-law fixed point nor a pure logarithm, but a log-corrected power-law regime. This is consistent with d = 2 being a marginal dimension.

6. Breakpoint scaling: finite-size boundary tail

The breakpoint position $r*$ (denoted `split_r`) was examined as a function of system size N. An extended bin-resolution test (bins up to 200, 294 additional runs) was used to determine the asymptotic scaling exponent x in $\text{split_r} \sim N^{x}$ by extrapolating x(bins) linearly in 1/bins toward bins → ∞.

EXTRAPOLATED BREAKPOINT EXPONENT X_∞

D	E	X_∞ (EXTRAPOLATED)	X AT BINS = 200	INTERPRETATION
d = 2	10^{-3}	0.855	0.799	approaches 1
d = 2	10^{-4}	0.956	0.931	consistent with $x \to 1$
d = 2	10^{-5}	0.993	1.031	consistent with $x \to 1$
d = 3	10^{-3}	0.935	0.902	consistent with $x \to 1$
d = 3	10^{-4}	1.015	1.010	consistent with $x \to 1$
d = 3	10^{-5}	0.959	0.938	consistent with $x \to 1$
d = 4	10^{-3}	1.059	0.947	approximately linear
d = 4	10^{-4}	1.066	0.982	approximately linear
d = 4	10^{-5}	1.061	0.982	approximately linear

INTERPRETATION

Across all dimensions and all physically reliable ε values, the breakpoint exponent converges toward x = 1, i.e. `split_r ~ N`. The breakpoint position is proportional to the system size with no additional scaling structure. This identifies the outer sector as a **finite-size boundary tail**: it moves linearly with the box, and would disappear in the infinite-volume limit.

The d = 4 extrapolations are slightly above 1, which may reflect mild extrapolation bias at the available N range or stronger boundary influence in higher dimensions. This does not alter the main conclusion: all values cluster near 1 and none support an independent sublinear or superlinear exponent.

7. Outer sector: a_2 stability analysis

If the outer sector were an independent physical regime with its own fixed-point exponent, a_2 would be stable with respect to N. The log-log slope of $a_2(N)$ was computed for all (d, ε) combinations.

D	MEAN LOG-LOG SLOPE OF $A_2(N)$	INTERPRETATION
d = 2	**−0.273**	a_2 decreases with N
d = 3	**−0.258**	a_2 decreases with N
d = 4	**−0.082**	mild decrease, still negative

The outer exponent is systematically N-dependent in all dimensions. As the system grows, the boundary-influenced region occupies a smaller fraction of the total radial range, and a_2 shifts accordingly. This behavior is inconsistent with an independently physical fixed-point regime, and is fully consistent with the finite-size boundary tail interpretation.

8. Bins robustness

Coefficient of variation (CV) over the bins parameter reveals a strong asymmetry between the inner and outer sectors.

QUANTITY	TYPICAL CV	INTERPRETATION
a_1 (inner exponent)	**0.01 - 0.08**	**Stable — bins-independent**
split_r / N (breakpoint ratio)	**0.07 - 0.15**	Moderate sensitivity
a_2 (outer exponent)	**0.09 - 1.45**	**High sensitivity; not physically robust**
AIC gain	**0.13 - 0.31**	Bins-dependent; interpret in relative terms

The inner exponent a_1 is strikingly stable across bins choices. This stability confirms that a_1 reflects a genuine property of the underlying field and not a discretization artifact. The outer exponent a_2 shows the opposite behavior —

its high and variable CV, particularly in the $d = 4$, $N = 128$, $\varepsilon = 0.01$ case (CV ≈ 1.45), further supports the boundary tail interpretation.

9. Summary and interpretation

ROBUST FINDINGS

- Single power law is globally insufficient
- Two-regime structure is real (168/168)
- Inner sector is the physical regime
- d=4: $a_\infty \approx 2.05 \approx d-2$ (within 2.5%)
- d=3: a_∞ converging toward $d-2 = 1$
- d=2: log-corrected power law (120/120)
- a_1 is bins-stable; a_2 is not
- a_2 decreases with N — not a fixed point
- Breakpoint scales as split_r ~ N
- Outer sector = finite-size boundary tail

OPEN QUESTIONS

- Theoretical derivation: why split_r ~ N?
- d=3: when does a_∞ fully converge to 1?
- d=2: precise form of the log-correction?
- d=4, $N > 128$: further confirmation of $a_\infty \to 2$
- Analytical understanding of $\varepsilon = 0.01$ anomaly
- d=5: does $a_\infty \to d-2 = 3$ continue?

The spectral Green function robustly decomposes into two regimes across all explored dimensions and system sizes. The breakpoint scales linearly with system size, identifying the outer regime as a finite-size boundary tail. The physically meaningful signal is the inner core regime: in $d = 4$ it approaches the classical $d - 2$ expectation to within ~2.5%, in $d = 3$ it shows a clear convergent trend toward the same target, and in $d = 2$ it is better described by a log-corrected power law, consistent with $d = 2$ being a marginal dimension.

10. Limitations and next steps

The d = 4, N = 160 configuration could not be completed within the available 51 GB memory budget. The bins → ∞ extrapolation assumes linearity in 1/bins, which is plausible but not mathematically guaranteed; alternative functional forms (quadratic, logarithmic) have not yet been tested. The $a_1(N)$ extrapolation uses a grid search over the decay exponent β, which identifies the best-fit value without providing a full uncertainty envelope. The exclusion of ε = 0.01 is empirically motivated but lacks a complete theoretical account.

RECOMMENDED NEXT STEPS

PRIORITY	TEST	COST	EXPECTED IMPACT
High	Dense N-series in d=3 (N = 32–160, ≥20 runs per N)	4–6 h	Tight a_∞ estimate and convergence rate
High	Alternative 1/bins extrapolation forms (quadratic, log)	~2 h	Validates x → 1 robustness
Medium	d=4, N = 192–256 if memory allows	10–20 h	Strongest possible a_∞ → 2 confirmation
Medium	Analytical derivation of split_r ~ N for a boundary tail	—	Would theoretically close the outer regime question
Low	d=5 extension	5–10 h	Tests generality of a_1 → d−2 pattern

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Total runs: 462 (168 main + 294 scaling test)

Memory budget: ~51 GB (Google Colab)

Generator: Regularized spectral Poisson kernel, $\varepsilon \in \{10^{-3}, 10^{-4}, 10^{-5}\}$